**My Mathematical Napkin**

A Collection of Important Notes

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# **Metamathematics**

## **Notational Conventions**

denotes a deduction[[1]](#footnote-1). is to be read as “ yields ” or follows from ”.

Capital Greek Letters, , etc.…, denote a finite sequence of assumption formulas[[2]](#footnote-2).

If we wish to emphasize that a certain variable or a certain set of variables, occurs in our finite sequence of assumption formulas, we use the notation or , as needed.

## **Introduction [TO-DO]**

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## **Introducing a Formal Metamathematical System**

Similar to a language, in the study of metamathematics we must first identify the various symbols and grammar rules that we will be operating on. In this study, we are interested in understanding how to manipulate these symbols in a formal manner and outside of any context or interpretation.

### **Definition** A symbol is a formal symbol of the metamathematical language if it is an occurrence of any of the following:

* A propositional connective, namely: (implies), (and), (or), (not).
* A quantifier, namely: (for all), (there exists).
* (equals)
* (addition), (multiplication), (successor i.e., if )
* (zero)
* (variables)
* and (parentheses).

### **Definition** A **formal expression,** or simply expressionis constructed from a finite sequence of occurrences of formal symbols

### **Example** The following are examples of formal expressions:

The metamathematical system also allows us to view formal expressions as strings that can be combined with each other by appending one to the other. This operation is called **concatenation.**

For example, we can concatenate the expressions and to produce the new expression:

Of course, some formal expressions, while valid, are still meaningless without the introduction of formation rules. In the usual, informal mathematics, for example, is nonsensical. To that end, we introduce the following definitions.

### **Definition** A **term** in the formal system represents the natural numbers (fixed or represented), specifically defined by the following inductive definition.

1. 0 is a term.
2. A variable is a term.
3. If and are terms, then is a term
4. If and t are terms, then is a term
5. If is a term, then is a term.
6. The only terms are those given by these rules.

### **Definition** A **formula** (or **well-formed formula)** is defined by the following inductive definition:

1. If and are terms, then is a formula.
2. If and are formulas, then is a formula.
3. If and are formulas, then is a formula.
4. If and are formulas, then is a formula.
5. If is a formula, then is a formula.
6. If is a variable and is a formula, then is a formula.
7. If is a variable and is a formula, then are formulas.
8. The only formulas are those given by these rules.

While we have defined terms and formulas with parentheses enclosing the operands, we will choose to omit these for the sake of readability, unless doing so introduces ambiguity. Additionally, for the sake of brevity, we introduce the following abbreviation rules:

is an abbreviation for

is an abbreviation for , where is a variable and and are terms not containing .

Aside from concatenation, we also have another operation on a term in our formal system, which we introduce below. But first, we need to characterize our variables based on where they are in a formula.

### **Definition** An occurrence of a variable in a formula is **bound** if the occurrence is in the scope of a quantifier or or is in a quantifier. Otherwise, is **free.** Binding pertains to the innermost quantifier in the formula.

In terms of interpretation, an expression containing a free variable represents a quantity or proposition dependent on the value of the variable. Otherwise, if the expression contains a bound variable, the expression represents the result of an operation performed over the range of the variable.

### **Definition** The **substitution** of a term for a variable in a term or formula consists of replacing simultaneously each free occurrence of by an occuernce in .

More formally, if we denote as the term with as a free variable (not necessarily in ), then is the substitution operation and for which are (possibly empty) parts not containing

The meaning of a formula is preserved when substitution is performed on a free variable. More specifically, this occurs when we substitute for , the term , in the formula where no free occurrence of occurs in a quantifier bound by a variable of . In such a case, is **free at the free occurrences of** .

### **Example** Let be the term . Then substitution for is valid for the first formula, but not the second

## **Introducing the Postulates**

Aside from the different symbols and formulae, we introduce a series of postulates for our formal system. These serve as the assumptions within our formal system that we take as true without question.

### **Axiomatic System**

For Postulates 1-8, are formulae.

For Postulates 9-13 is a variable, is a formla is a formula which does not contain free, and is a term which is free for in

**Postulates for the propositional calculus**

Postulate 1a:

Postulate 1b:

Postulate 2: ,

Postulate 3:

Postulate 4a:

Postulate 4b:

Postulate 5a:

Postulate 5b:

Postulate 6:

Postulate 7:

Postulate 8:

**Additional Postulates for the predicate calculus**

Postulate 9:

Postulate 10:

Postulate 11:

Postulate 12:

**Additional Postulates for Number Theory**

Postulate 13:

Postulate 14:

Postulate 15:

Postulate 16:

Postulate 17:

Postulate 18:

Postulate 19:

Postulate 20:

Postulate 21:

These postulates serve as a template for various axioms.

### **Definition** A formula is an **axiom** if it is one of the forms 1a, 1b, 3—8,10, 11, 13 or if it is of the form 14—21.

Postulates 2, 9, and 12 are **rules of inference,** and give a notion of immediate consequence within our formal system. Namely, for these postulates the last statement is a **conclusion** and the preceding statements are **premises.**

Axioms combined with rules of inference are used to construct proofs, which we also formalize below.

### **Definition** A formula is **provable** as defined inductively below:

1. If is an axiom, then is provable.
2. If is provable, and is an immediate consequence of , then is provable.
3. If and are provable, and is an immediate consequence of and , then is provable.
4. A formula is provable only as required by 1—3.

**Definition** A **formal proof** is a finite sequence of one or more occurrences of formulas such that each formula of the sequence is either an axiom or an immediate consequence of preceding formulas of the sequence. It is said to be the proof of the last formula in the sequence.

### **Example** The following is an example of a proof within the formal system:

Prove:

1. By Postulate 1a.
2. By Postulate 1b.
3. By Postulate 2, 1, 2
4. By Postulate 1a.
5. By Postulate 2, 3, 4

## **Formalizing Deduction**

Having formally described the metamathematical language, we now seek to apply it to prove various theorems. In theory, we would use only axioms and rules of inference for our proofs, however in practice, and indeed in informal mathematics, we often abbreviate. The simplest form of abbreviation is through the use of **derived theorems,** those which follow directly from the application of axioms and rules of inference. Reasoning in this manner is referred to as **direct deduction** Of course, some proof techniques such as proof by contradiction require that we set up assumptions of our own outside of the given axiomatic system. Reasoning in this manner is called **subsidiary deduction.**

We now seek to generalize our definition of proofs[[3]](#footnote-3) to apply to deductions as well.

### **Definition** Given a list , , of occurrences of formulas, referred to as **assumption formulas,** a finite sequence of one or more occurrences of the formulas is called a **formal deduction** if each formula in the sequence is either one of the assumption formulas, or an axiom, or an immediate consequence of preceding formulas in the sequence. It is said to be a **deduction** of the last formula in the sequence.

### **Definition** Let , be a list of assumption formulas and be the last formula of some formal deduction under the assumption formulas. Then, is **deducible** from the assumption formulas, and is referred to as the **conclusion** of the deduction. In symbolic form this is written as:

These definitions allow for the use of any assumption formulas outside of the given postulates in our Axiomatic System[[4]](#footnote-4).

**Example** The following is an example of a deduction. Our desired conclusion is

1. Assumption Formula 1
2. Assumption Formula 2
3. Assumption Formula 3
4. Postulate 3
5. Postulate 2, 1, 4
6. Postulate 2, 2, 5
7. Postulate 2, 3, 6

Notice that at each step of the deduction, we write what a brief explanation as to why the formula follows. We refer to such an explanation as an **analysis** of the deduction.

In the above example, we proved that follows from the assumption formulas. In line with the notation introduced in Definition 3.2., we may write our conclusion as:

We now seek to characterize deduction as a sequence. The following listing shows some of these properties.

### **Listing .** Given lists of assumption formulas , assumption formulas and , and conclusion

1. If , then is in the list
2. If , then

(Extraneous assumptions do not affect the conclusion if it is deducible).

1. If , then

(The order of assumptions does not affect the conclusion if it is deducible).

1. If , and , then

(Assumptions which are derived from previous assumptions can be rederived in the deduction as needed without affecting the conclusion).

An important result within Metamathematics is the following Theorem. It allows us to verify that abbreviating proofs is, indeed, valid.

### **Theorem** . **(The Deduction Theorem for Propositional Calculus)**

For the propositional calculus, if , then

In other words, if is deducible from , then there exists a deduction using where the final element in the sequence is .

**Idea of the Proof:**

Show the following is true by strong induction:

*: For every formula , if there is a deduction of from of length , then there can be found a deduction of from*

Consider the following base cases and show that follows from :

1. is one of the formulas of
2. is .
3. is an axiom.

In addition to the above cases, for the inductive case, consider when is the consequence of two preceding formulas in the deduction applying Postulate 2 (Modus Ponens). Namely, there exists and .

If we apply the Deduction Theorem to the metamathematical statement to obtain , we say that we **discharge** . The resulting metamathematical statement is referred to as a direct type since we only apply axioms and rules of inference on our assumptions to arrive at the conclusion.

Earlier, we introduced the notion of a subsidiary deduction. We formalize such a deduction through the definition below

### **Definition** A **subsidiary deduction rule** is a metamathematical theorem which has one or more hypotheses of the form and a conclusion of the form . Each of the are **subsidiary deductions**, and the conclusion is the **resulting deduction.**

**Example** The following is an example of a subsidiary deduction rule:

If and , then .

and are the subsidiary deductions, and is the resulting deduction.

# **Elementary Functions**

## **What are functions?**

This chapter serves to catalog important functions that are relevant in different mathematical theories, and which merit their own section to explore the different properties of these functions.

To begin, we formally define what we mean by a function

### **Definition** A **function** is a mapping between two sets and that associates each element in to one element in . We denote it as .

### **Definition** Given a function , the set is referred to as the **domain,** and the set is referred to as the **codomain.**

We denote to be an element of , and to be an element of such that (read as maps to ).

### **Definition** Given a function , the **range** of is defined as the set . It is the set of possible values within the codomain that are mapped to. The range and the codomain are not necessarily the same.

## **Absolute Value**

The absolute value function is a special function, defined as , where

Geometrically, the absolute value function represents the unsigned distance between a number on the real number line, and .

We now examine some important properties of the absolute value function. For the following, we let .

### **Proposition** .

**Proof:** This should be apparent if we do an exhaustive proof considering the four cases when is positive or negative, and is positive or negative.

### **Proposition** .

**Proof:** The intuitive notion of the absolute value representing distance between two numbers should make this trivial. For formality’s sake, we can use 2.2.1 and express as

### **Proposition** **(Triangle Inequality)** .[[5]](#footnote-5)

**Proof:** We can do an exhaustive proof, considering the four cases when is positive or negative, and is positive or negative.

### **Proposition** **(Reverse Triangle Inequality)** .

**Proof:** Express and as and , respectively, and apply 2.2.2 and the Triangle Inequality to obtain an inequality with an upper and lower bound. Conclude based on the intuitive interpretation of that the proposition hold.

### **Proposition** .

**Proof:** It follows directly from applying the Triangle Inequality to:

# **Real Analysis**

## **Notational Conventions**

We assume the use of standard mathematical notation.

## **Motivation for**

In real world applications such as Physics, we often deal with quantities whose values span a continuum. In that sense, it no longer suffices to use , , or even as the domain in which to define our values since doing so introduces discontinuities in our supposedly continuous domain. This shall serve as our primary, application-oriented, motivation for introducing a superset of the rationals that can “fill in the gaps” between the rational numbers. Such a set is what we refer to as the set of real numbers or .

1. See 1.5.2 [↑](#footnote-ref-1)
2. See 1.5.1 [↑](#footnote-ref-2)
3. See 1.4.3 [↑](#footnote-ref-3)
4. See 1.4.1 [↑](#footnote-ref-4)
5. As it turns out, this inequality has a geometric interpretation if we imagine as the hypotenuse of a right triangle with legs of length and . If the triangle is indeed a triangle in Euclidean space, then the Triangle Inequality must be satisfied. [↑](#footnote-ref-5)